

AXIALLY SYMMETRIC FLOW IN TWO SPACE DIMENSIONS FOR NONIDEAL GASES

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Abstract: We study axially symmetric Euler's system of equations for the real gas in two space dimensions. This problem poses various difficulties for arbitrary equation of state and general problem is still open. Deviation from ideal gas law generally involves change of phase and more number of chemical species. This affects number of degrees of freedom which is governed by phase rule. We analyze self-similar solutions of Euler's system of equations for the real gas (alternatively called non-ideal gas) which is in one phase but may contain vapor or moisture. We find that structure of solutions in this case is governed by fundamental derivative of underlying gas. Emergence of fundamental derivative in calculations carried out by the authors suggest that structure of solutions to Euler's equations in gas dynamics is fundamentally dependent on the thermodynamics of underlying gas. We have considered case of zero swirl in this article.

Keywords and Phrases: Euler's equations, Axially symmetric Flow, Fundamental Derivative, Mach number, Swirl, Gas dynamic Waves, Transitional Solution, Chaos.

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1. Introduction

Method of characteristics is central in dealing with hyperbolic system of partial differential equations in one space dimension and widely discussed in the literature. However characteristic surfaces admitted by hyperbolic system of partial

differential equations in several space variables is less studied because of complicated geometries displayed by characteristic surfaces of these systems. Problem of Euler's system of equations for arbitrary equation of state is open. Ruggeri [1] [3] has investigated a model for nonideal gases. We have undertaken exercise of exploring possibilities this model of non ideal gas and Euler's equations under the assumption of axisymmetry. We have found that solutions in this case are determined by so called fundamental derivative. In the first part we will list a few properties of fundamental derivative.

2. Equation of State and Fundamental Derivative

One of the most important requirement to study gas dynamics is that underlying system remain hyperbolic. Convexity of the equation of state is another requirement that limits the description of wave motion in real gases. Van der Waals equation models real gases with a sufficient accuracy and also takes into account phase change. However van der Waals equation is not convex in ρ and this makes study of solutions of Euler's system of equations for real gases really difficult. Van der Waals model takes into account phase change which essentially requires correction to pressure and volume in ideal gas law. One of the method in deriving van der Waals gas from ideal gas is method of virial expansion and it leads, upon neglecting correction to pressure term, to the model for real gases which is convex function of ρ . Model so obtained is also given in Wu et. al. [2]. This equation of state is used extensively by Ruggeri for the development of Extended Rational Thermodynamics [1]. It is this model that this article concerns with and it is stated below.

$$p(\rho) = A \frac{\rho^\gamma}{(1 - \rho b)^\gamma} \quad (1)$$

Interestingly this model takes into account presence of moisture. Because of presence of moisture, one chemical component increases in the gas and it amounts to addition of one degree of freedom to those with ideal gas. Phase rule then requires one more intensive variable to explain dynamics of flow. We propose that additional intensive variable sought is fundamental derivative for underlying real gas and this article deals with how fundamental derivative explains gas dynamic waves in case of axially symmetric flow described by Euler's system of equations. Note that change of phase contributes to entropy production, here no change of phase takes place and only presence of moisture is taken into account. Fundamental derivative is omnipresent in the literature of real gases. Thompson [4] has discussed properties of fundamental derivatives and noted also behavior of gas when $\Gamma < 0$. For real gases it is found that when $\Gamma > 0$ then actually $\Gamma > 1$. One of the fact about fun-

damental derivative is that for $\Gamma > 0$ sonic point corresponds to maximum entropy where as for $\Gamma < 0$ sonic point corresponds to point of minimum entropy. In gas dynamics the case $\Gamma < 0$ results in anomalous behavior. In the notations used in the present article we use the following expression for fundamental derivative.

$$\Gamma = \frac{1}{\sqrt{p'}} \frac{\partial \rho \sqrt{p'}}{\partial \rho} \quad (2)$$

It is well known in the literature that existence and stability of compression shocks in gases is closely related to the condition

$$\frac{\partial^2 V}{\partial P^2} > 0. \quad (3)$$

If this condition is violated then it is observed that rarefaction shocks are stable. This quantity is not in non dimensional form. Using this derivative a non dimensional quantity is introduced in the literature and this quantity is known as fundamental derivative [4]. It is defined as follows

$$\Gamma = \frac{c^4}{2V^2} \left(\frac{\partial^2 V}{\partial P^2} \right). \quad (4)$$

For ideal gas with constant specific heats we have

$$\Gamma = \frac{1}{2}(\gamma + 1) \quad (5)$$

where γ is adiabatic gas constant.

In ideal gas it was found that solutions depend on adiabatic exponent γ . In the considered model so called fundamental derivative emerge as a quantity which govern the solutions. For this equation of state we have the following expression for fundamental derivative

$$\Gamma = 1 + \frac{1}{2}((\gamma - 1) + \frac{(\gamma + 1)b\rho}{1 - b\rho}). \quad (6)$$

3. Main Equations

Initial part of this section essentially borrowed from Zheng [5]. This article deals with the equation of state (1). We consider the two dimensional isentropic Euler equations

$$\begin{aligned} \rho_t + (\rho u)_x + (\rho v)_y &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0. \end{aligned} \quad (7)$$

We assume axisymmetry; that is, we assume that solutions (ρ, u, v) have the property:

$$\begin{aligned}\rho(t, r, \theta) &= \rho(t, r, 0), \\ \begin{pmatrix} u(t, r, \theta) \\ v(t, r, \theta) \end{pmatrix} &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} u(t, r, 0) \\ v(t, r, 0) \end{pmatrix}.\end{aligned}\quad (8)$$

By considering polar coordinates (r, θ) of the (x, y) plane we get the following system of balance laws.

$$\begin{aligned}\rho_t + (\rho u)_r + \frac{\rho u}{r} &= 0, \\ (\rho u)_t + (\rho u^2 + p)_r + \frac{\rho(u^2 - v^2)}{r} &= 0, \\ (\rho v)_t + (\rho uv)_r + \frac{2\rho uv}{r} &= 0.\end{aligned}\quad (9)$$

Self similar analysis of (9) leads to the following equations where $I = su$ and $K = sv$. [6]

$$I = su \text{ and } K = s\sqrt{p'(\rho)} \quad (10)$$

Thus under the condition of axial symmetry and zero swirl, we have the following initial data-

$$\rho(0, r, 0) = \rho_0, \quad u(0, r, 0) = u_0, \quad v(0, r, 0) = v_0 = 0. \quad (11)$$

Note that Γ is non-dimensional quantity and needed no transformation or change of variable. After carrying out several manipulations we get the following system of equations.

$$\begin{aligned}\frac{dK}{d\tau} &= K(K^2 - (1 - I)^2) - KI(\Gamma - 1)(1 - I), \\ \frac{dI}{d\tau} &= K^2I - I(K^2 - (1 - I)^2), \\ \frac{d\Gamma}{d\tau} &= (\gamma + 1)I(1 - I)\frac{2\Gamma(2\Gamma - \gamma - 1)}{(1 + \gamma)^2}, \\ \frac{ds}{d\tau} &= -s(K^2 - (1 - I)^2).\end{aligned}\quad (12)$$

Equations (12) form an autonomous subsystem with polynomial right hand sides. Corresponding to initial data (11) we will attempt for analyzing solutions with the following initial condition

$$(I, K, \Gamma) = (s(u_0, \sqrt{p'(\rho_0)}, \Gamma_0) \text{ when } \lim s \rightarrow 0+ \quad (13)$$

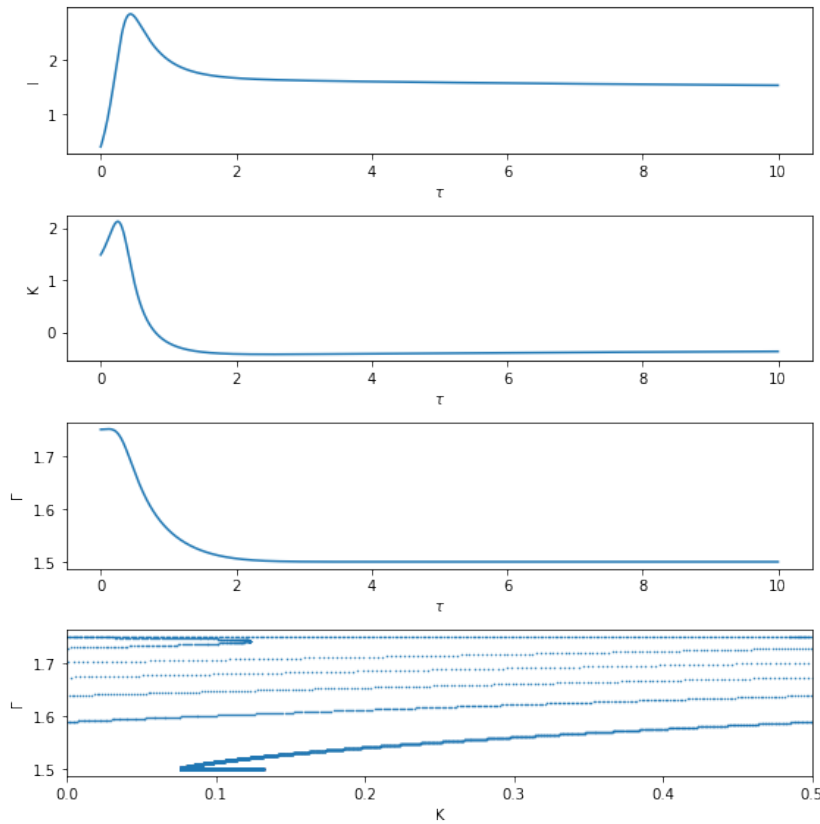


Figure 1: Near Chaotic Behavior of Γ against K at Transitional Solution.

4. Analysis and Discussion

Computations on which our analysis is based upon are all given separately in appendix. Notable is the behavior of system of ordinary differential equations (12) in the neighborhood of equilibrium point (synonymous to stationary point of a dynamical system represented by system of ordinary differential equations). This equilibrium point gives rise to, so called, transitional solution asymptotically as self similar parameter goes to infinity. At this point $I = 1/\gamma$ and $K = \frac{\gamma^2-1}{\sqrt{(2)\gamma}}$. At equilibrium position $\Gamma = \frac{\gamma+1}{2}$. Our calculations suggest that Γ shows no change in its behavior, in fact remains very close to a constant, near all equilibrium point except at the equilibrium point giving rise to transitional solution. Authors believe that at transitional point, which we have denoted by Q , system (12) shows chaotic behavior. To support our claim we are giving the following diagrams. If one disregards presence of moisture and treats dry gas then system's behavior is very

simple. In that case $I - K$ plane is divided into two invariant regions separated by separatrices for the underlying system. When we depart from ideal gas law even a smaller (that is consider the equation of state (1)) then behavior changes drastically near Q . In this case Γ no more remains constant or near constant but shows variation in its values in comparison with variation in K . In this case there are no separating regions which control flow variables in respective regions but integral curves of the system become near chaotic. Details investigations, theoretical and numerical will consume a few more articles and here we limit ourselves to demonstrate our findings by carefully drawn diagrams.

4.1. Transitional Solution

We want to estimate now the data (u_0, ρ_0) which leads to the transitional solution. Transitional solution corresponds to the stationary point Q . We introduce $L := (\frac{K}{I})^2$. Upon simplification we get the following ordinary differential equation for L . L is dimensionless quantity like M^2 .

$$\frac{dL}{dI} = \frac{2L}{I} \frac{-LI^2 + (1-I)^2 - (\Gamma-1)I(1-I) + 2LI^2 - (1-I)^2}{-2LI^2 + (1-I)^2}, \quad (14)$$

$$L = M_0^{-2}. \text{ at } I = 0 \quad (15)$$

We want to estimate on the lines of Zheng [6] an initial condition which leads to the transitional solution given by $Q = (\frac{1}{2\Gamma-1}, 2(\Gamma-1)^2)$ Upper bound for initial Mach number is $\frac{1}{\sqrt{2(\Gamma-1)}}$. Solution starting at such Mach number starts as a decreasing function of $I > 0$ till $I = \frac{1}{2\Gamma-1}$. Subsequently solution remains below of the two curves defined by numerator and denominator of (14) and therefore remains decreasing until the solution reaches stationary point $(I, L) = (1, 0)$. As it is the case with ideal gas, in this case also solution approaches the point Q as $s \rightarrow \infty$ since at this point $(1-I)^2 - K^2 \neq 0$ and we get transitional solution as $r \rightarrow 0$ In this case we get the following solution:

$$u(r) = \frac{1}{2\Gamma-1}r, \quad p'(\rho) = \frac{2(\Gamma-1)^2}{(2\Gamma-1)^2}r^2. \quad (16)$$

This is an asymptotic form of the solution when $r \rightarrow 0$. This situation is described in 1.

5. Conclusion

Emergence of fundamental derivative while analyzing solutions of axially symmetric flow is an important observation. In the light of dependence of the structure of solutions on the thermodynamics of underlying gas, we have found that this dependence is owing to presence of moisture or vapor in the gas thereby increasing

number of components due to phase rule. Authors believe that there must be some way to make this dependence explicit in case of arbitrary equation of state. Authors also believe that transitional solution shows chaotic behavior. Theoretical investigation of existence (or nonexistence) of strange attractor for the underlying dynamical system will shade some light on the mechanics of multi component gases.

6. Appendix

6.1. Linearization

6.1.1. Linearization at $(0, 0, 0)$

(12) linearizes to the following at $(0, 0, 0)$

$$\begin{pmatrix} \frac{dI}{d\tau} \\ \frac{dK}{d\tau} \\ \frac{d\Gamma}{d\tau} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I \\ K \\ \Gamma \end{pmatrix} + \text{second order terms} \quad (17)$$

6.1.2. Linearization at $(0, 0, (1 + \gamma)/2)$

(12) linearizes to the following at $(0, 0, (1 + \gamma)/2)$

$$\begin{pmatrix} \frac{dI}{d\tau} \\ \frac{dK}{d\tau} \\ \frac{d\Gamma}{d\tau} \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ -\frac{\gamma+3}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I \\ K \\ \Gamma - \Gamma_0 \end{pmatrix} + \text{second order terms.} \quad (18)$$

6.1.3. Linearization at $(1, 0, (1 + \gamma)/2)$

(12) linearizes to the following at $(1, 0, (1 + \gamma)/2)$

$$\begin{pmatrix} \frac{d(I-1)}{d\tau} \\ \frac{dK}{d\tau} \\ \frac{d\Gamma}{d\tau} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\gamma-1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (I-1) \\ K \\ \Gamma - \Gamma_0 \end{pmatrix} + \text{second order terms.} \quad (19)$$

6.1.4. Linearization at $(0, 1, (1 + \gamma)/2)$

(12) linearizes to the following at $(0, 1, (1 + \gamma)/2)$

$$\begin{pmatrix} \frac{d(I-1)}{d\tau} \\ \frac{dK}{d\tau} \\ \frac{d(\Gamma-\Gamma_0)}{d\tau} \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ \frac{\gamma+3}{2} & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (I-1) \\ K \\ \Gamma - \Gamma_0 \end{pmatrix} + \text{second order terms.} \quad (20)$$

6.1.5. Linearization at Q

(12) linearizes to the following at Q

$$\begin{pmatrix} \frac{d(I-a)}{d\tau} \\ \frac{d(K-b)}{d\tau} \\ \frac{d\Gamma}{d\tau} \end{pmatrix} = \begin{pmatrix} L & M & 0 \\ N & P & \gamma-1 \\ 0 & 0 & \frac{2(\gamma-1)}{\gamma^2} \end{pmatrix} \begin{pmatrix} I-a \\ K-b \\ \Gamma - \Gamma_0 \end{pmatrix} + \text{second order terms} \quad (21)$$

$$\begin{aligned}
 a &= \frac{1}{\gamma}, & b &= \frac{(\gamma^2 - 1)\sqrt{2}}{\gamma}, \\
 N &= \gamma^3 + \gamma^2 - 2\gamma, \\
 M &= 2\sqrt{2}\frac{\gamma^2 - 1}{\gamma}, \\
 L &= 2(\gamma - 1), \\
 P &= \sqrt{2}\frac{\gamma^2 - 1}{\gamma}.
 \end{aligned}$$

6.1.6. Linearization at $(1, 0, 0)$

Linearization of (12) at $(1, 0, 0)$ has zero linear part.

$$\begin{pmatrix} \frac{d(I-1)}{d\tau} \\ \frac{dK}{d\tau} \\ \frac{d\Gamma}{d\tau} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (I-1) \\ K \\ \Gamma \end{pmatrix} + \text{second order terms.} \quad (22)$$

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